Weak square from weak presaturation

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Motivating questions

Question

Are "hugeness" and "compactness" properties of ω_2 simultaneously consistent? More specifically, can there exist a saturated ideal on ω_2 while ω_2 has the tree property?

Question (Foreman-Magidor)

Can there exist a saturated normal ideal on ω_2 that contains the approachability ideal?

We show that the answer to both questions is negative if $2^\omega \leq \omega_2$.

Theorem

Suppose κ is a regular cardinal, $2^{<\kappa} \leq \kappa^+$, and there is a weakly presaturated ideal on κ^+ concentrating on $\mathrm{cof}(\kappa)$. Then $\square^*_\kappa$ holds.

Definitions: Ideals

A normal ideal I on a regular cardinal is:

1. *precipitous* if whenever $G \subseteq \mathcal{P}(\kappa)/I$ is generic, the ultrapower $\mathrm{Ult}(V, G)$ is well-founded.
2. *saturated* if $\mathcal{P}(\kappa)/I$ has the κ^+-c.c.
3. *presaturated* if for every I-positive set S and every set of antichains $\{\mathcal{A}_\alpha : \alpha < \mu\}$, where $\mu < \kappa$, there is an I-positive $S' \subseteq S$ such that for each $\alpha < \mu$, $|\{T \in \mathcal{A}_\alpha : T \cap S' \notin I\}| \leq \kappa$.
4. *strong* if I is precipitous and it is forced that $j_G(\kappa) = (\kappa^+)^V$.
5. *weakly presaturated* if it is forced that $\mathrm{Ult}(V, G)$ is well-founded up to $(\kappa^+)^V + 1$ and $j_G(\kappa) = (\kappa^+)^V$.

If κ is a successor cardinal, then (2)–(5) are successively weaker properties. Well-foundedness up to $(\kappa^+)^V$ always holds.

Definitions: Trees and weak square

For an infinite cardinal κ, a *κ-tree* is a tree of height κ with levels of size $< \kappa$. We say that κ has the *tree property* if every κ-tree has a cofinal branch. A κ^+-tree is called *special* if there is a function $f : T \to \kappa$ such that $x < y$ implies $f(x) \neq f(y)$. Clearly, special κ^+-trees cannot have cofinal branches, since a branch would witness that κ^+ is not a cardinal.

Jensen showed the existence of a special κ^+-tree is equivalent to the weak square principle $\square^*_\kappa$, which states that there is a sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa^+ \rangle$ such that:

1. Each $\mathcal{C}_\alpha$ is a nonempty set of size $\leq \kappa$ consisting of closed unbounded subsets of α, each of order-type $\leq \kappa$.
2. If $\alpha < \kappa^+$, $D \in \mathcal{C}_\alpha$, and $\beta < \alpha$ is a limit point of D, then $D \cap \beta \in \mathcal{C}_\beta$.

Definitions: Approximation and prewellordering

Suppose A is a set, $X \subseteq A$, and $\mathcal{F} \subseteq \mathcal{P}(A)$. We will say that X *is approximated by* $\mathcal{F}$ when for all $a \in \mathcal{F}$, $a \cap X \in \mathcal{F}$.

For models of set theory $M \subseteq N$ and an M-cardinal κ, the pair (M, N) satisfies the κ-approximation property when for all ordinals $\lambda \in M$, if $X \in \mathcal{P}(\lambda)^N$ is approximated by $\mathcal{P}_\kappa(\lambda)^M$, then $X \in M$.

A *prewellordering* is a transitive reflexive binary relation in which every two elements are comparable, such that the quotient by the equivalence relation, $x \sim y \Leftrightarrow x \leq y \wedge y \leq x$, is a wellorder.

There is a natural correspondence between surjections from sets onto ordinals and prewellorderings of those sets. For a set of ordinals Z closed under the Gödel pairing function, a set $X \subseteq Z$ codes a relation on Z via this function. If $X \subseteq Z$ codes a prewellordering whose quotient has order-type α, let $f_X : Z \to \alpha$ be the corresponding surjection.

Proof of the theorem

Theorem

Suppose κ is a regular cardinal, $2^{<\kappa} \leq \kappa^+$, and there is a weakly presaturated ideal on κ^+ concentrating on $\mathrm{cof}(\kappa)$. Then $\square^*_\kappa$ holds.

We may assume that $\kappa > \omega$. Let $\delta = (\kappa^+)^V$. A forcing introduces an elementary embedding $j : V \to M$ with critical point δ, such that M is well-founded up to $(\kappa^{++})^V + 1$, $j(\delta) = (\kappa^{++})^V$, and $M \models \mathrm{cf}(\delta) = \kappa$.

We will construct a $\square^*_\kappa$-sequence in M. By elementarity, $\square^*_\kappa$ will hold in V.

Set-up and key claim

Since $V \models \delta^{<\kappa} = \delta$, $\mathcal{P}_\kappa(\delta)^V \in M$. Define in M the set $\mathcal{A}$ of subsets of δ that are approximated by $\mathcal{P}_\kappa(\delta)^V$.

Fix in M a club $C^* \subseteq \delta$ of order-type κ, and let $\langle \xi_\alpha : \alpha < \kappa \rangle$ be its increasing enumeration. In V, let $\vec{\sigma} = \langle \sigma_\alpha : \alpha < \delta \rangle$ be a sequence such that $\sigma_\alpha : \kappa \to \alpha$ is a surjection, and note that $\vec{\sigma} \in M$.

We can write δ as the union of a continuous increasing sequence of sets of size $< \kappa$, $\langle z_\alpha : \alpha < \kappa \rangle$, by putting $z_\alpha = \bigcup_{\beta < \alpha} \sigma_{\xi_\beta}[\alpha]$.

Take $N \prec H^M_{j(\delta)}$ such that $\{C^*, \mathcal{P}_\kappa(\delta)^V, \vec{\sigma}\} \cup \delta \subseteq N$ and $M \models |N| = \kappa$. Let $Q = \mathcal{P}_\kappa(\delta) \cap N$.

Claim 1

Suppose $X, Y \in \mathcal{A}$ code prewellorderings of δ of the same length. Then $\{f_X[z] : z \in Q\} = \{f_Y[z] : z \in Q\}$.

Range restriction

If f is a function from Z to an ordinal α and $\beta < \alpha$, let $f \downharpoonright \beta$ be the function g such that $g(\gamma) = f(\gamma)$ when $f(\gamma) < \beta$ and $g(\gamma) = 0$ otherwise.

If R is a prewellordering on a set Z of order-type α and $\beta < \alpha$, then let $R \downharpoonright \beta$ denote the canonical alteration of R to represent $f_R \downharpoonright \beta$.

Claim 2

If $X \in \mathcal{A}$ codes a prewellordering of order-type α and $\beta < \alpha$, then $X \downharpoonright \beta \in \mathcal{A}$. Furthermore, if $r \in Q$, then $f_X[r] \cap \beta = f_{X \downharpoonright \beta}[s]$ for some $s \in Q$.

Claim 3

For $\alpha < \delta^+$, $V \models \mathrm{cf}(\alpha) < \kappa$ iff $M \models \mathrm{cf}(\alpha) < \kappa$.

Construction of the weak square sequence

First consider ordinals $\alpha < j(\delta)$ of cofinality $< \kappa$. Let $\mathcal{C}_\alpha$ be the set of all clubs D in α of order-type $< \kappa$, such that for some $X \in \mathcal{A}$ that codes a prewellordering of δ of order-type α, $D = f_X[s]$ for some $s \in Q$.

By Claim 1, the choice of X does not matter, so the cardinality of this set is at most $|N| = \kappa$. By Claim 2, if $C \in \mathcal{C}_\alpha$ and β is a limit point of C, then $C \cap \beta \in \mathcal{C}_\beta$.

Claim 3 ensures that each such $\mathcal{C}_\alpha$ is nonempty.

For ordinals of cofinality, κ, we have two cases:

Case 1: $V \models \mathrm{cf}(\alpha) = \kappa$. Let $D \in V$ be a club in α of order-type κ. Let $f : \delta \to \alpha$ be a surjection in V. If s is an initial segment of D of limit order-type, then $r = f^{-1}[s] \in V$. If $\beta = \sup(s)$, then $s = (f \downharpoonright \beta)[r]$, so $s \in \mathcal{C}_\beta$.]

Case 2: $V \models \text{cf}(\alpha) = \delta$. Let $D \in V$ be a club in α of order-type δ, and let $\langle \gamma_\beta : \beta < \delta \rangle$ be its increasing enumeration. Let $f : \delta \to \alpha$ be a surjection in V. Let $g : \delta \to \delta$ be a function in V such that for all $\beta < \delta$, $f \circ g(\beta) = \gamma_\beta$.

In M, let $D' = \{\gamma_\beta : \beta \in C^*\}$. Let s be an initial segment of C^*. Let $\gamma = \sup(s)$, and let $\beta < \kappa$ be such that $s \subseteq z = \sigma_\gamma[\beta]$.

$g \restriction z$ is coded by an element of $\mathcal{P}_\kappa(\delta)^V$, and so $g \restriction s \in N$. Thus $\{\gamma_\beta : \beta \in s\} = f[r]$ for some $r \in N$.

In both cases, M has a club $C \subseteq \alpha$ of order-type κ such that all initial segments are in $\mathcal{C}_\beta$ for some $\beta < \alpha$. Let $\mathcal{C}_\alpha = \{C\}$ for any such C.

Semiproper ideals and CH

Corollary

Suppose I is a normal presaturated ideal on ω_2 and $\mathcal{P}(\omega_2)/I$ is a semiproper forcing. Then the continuum hypothesis holds.

The principle $\mathrm{SCC}^{\mathrm{cof}}$ states that for every large enough cardinal θ, every countable $M \prec H_\theta$, and every $\alpha < \omega_2$, there is a countable $N \prec H_\theta$ such that $M \subseteq N$, $M \cap \omega_1 = N \cap \omega_1$, and $\sup(N \cap \omega_2) > \alpha$. The proof just combines:

1. If $\mathcal{P}(\omega_2)/I$ is semiproper, then $\mathrm{SCC}^{\mathrm{cof}}$ holds. (Sakai '05)
2. If $\mathrm{SCC}^{\mathrm{cof}}$ holds, then the failure of CH is equivalent to the tree property at ω_2. (Torres-Perez and Wu '15)
3. $\mathrm{SCC}^{\mathrm{cof}}$ implies $2^\omega \leq \omega_2$. (Todorčević '91)
4. A presaturated ideal on κ^+ concentrates on $\{\alpha : \mathrm{cf}(\alpha) = \mathrm{cf}(\kappa)\}$. (Shelah '82)

Dim prospects for conventional forcing

The hypotheses below generalize constructions of saturated ideals from almost-huge cardinals. Note there is no assumption about cardinal arithmetic.

Theorem

Suppose $j : V \to M$ is an elementary embedding with critical point κ definable from parameters in V. Suppose $\mathbb{P} * \dot{\mathbb{Q}}$ is a two-step iteration such that:

1. *M is $|\mathbb{P}|$-closed, and $|\mathbb{P}| < j(\kappa)$.*
2. *$\mathbb{P} * \dot{\mathbb{Q}}$ collapses all ordinals in the open interval $(\kappa, j(\kappa))$.*
3. *Whenever $G * H$ is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V, then in some outer model, j can be lifted to $j' : V[G * H] \to M[G' * H']$, such that $\mathcal{P}_\kappa(\mathrm{Ord})^{V[G*H]} \subseteq M[G']$.*

Then $\mathbb{P}$ forces that $\kappa = \mu^+$ for some $\mu < \kappa$, and $\square^*_\mu$ holds.

Conclusion

It looks like we need new methods to answer the question of whether saturated ideals on ω_2 are compatible with the tree property.

Thanks for your attention!

Appendix: Proof of Claim 1

Let $r \in Q$. We need to show that there is some $s \in Q$ such that $f_X[r] = f_Y[s]$. There is a club $C \subseteq \kappa$ such that for all $\alpha \in C$, $f_X[z_\alpha] = f_Y[z_\alpha]$. We may assume that for all $\alpha \in C$, z_α is closed under Gödel pairing.

Let $\alpha \in C$ be such that $r \subseteq z_\alpha$. By definition, $z_\alpha \subseteq \xi_\alpha < \delta$. Since $|z_\alpha| < \kappa$, there is $\beta < \kappa$ such that $z = \sigma_{\xi_\alpha}[\beta] \supseteq z_\alpha$.

Since $X, Y \in \mathcal{A}$, $X \cap z$ and $Y \cap z$ are in V. Thus $X \cap z_\alpha$ and $Y \cap z_\alpha$ are in N. These sets code prewellorderings of z_α of order-type $\eta = \mathrm{ot}(f_X[z_\alpha]) = \mathrm{ot}(f_Y[z_\alpha])$.

Let $h_X : z_\alpha \to \eta$ and $h_Y : z_\alpha \to \eta$ be the corresponding surjections. Let $r' = h_X[r]$. Note that if $\pi : \eta \to f_X[z_\alpha]$ is the unique order-preserving map, then $\pi[r'] = f_X[r]$.

Let $s = h_Y^{-1}[r']$. Then $s \in N$, and $h_Y[s] = r'$. Furthermore, $\pi \circ h_Y[s] = f_Y[s] = f_X[r]$, as desired.