Higher Stationary Reflection and Cardinal Arithmetic

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Section 1

Higher Stationary Reflection

Stationary Reflection Principle

Definition (Stationary Reflection)

Let SR_{ω_1} be the following stationary reflection principle:

For any set $W \supseteq \omega_1$ and any stationary $X \subseteq \mathcal{P}_{\omega_1}(W)$, there is $R \subseteq W$ such that

- $|R| = \omega_1 \subseteq R$,
- $X \cap \mathcal{P}_{\omega_1}(R)$ *is stationary in* $\mathcal{P}_{\omega_1}(R)$.

- SR_{ω_1} is often called the Weak Reflection Principle (WRP).

Theorem (Foreman-Magidor-Shelah)

1. SR_{ω_1} holds if a supercompact cardinal is Lévy collapsed to ω_2.
2. Martin's Maximum implies SR_{ω_1}.

Consequences of SR_{ω_1}

SR_{ω_1} is known to have many interesting consequences.

Theorem ((1)–(3) Foreman-Magidor-Shelah, (4) Todorčević, (5) Shelah)

SR_{ω_1} implies the following.

1. Chang's Conjecture
2. NS_{ω_1} is presaturated.
3. All ω_1-stationary preserving posets are semi-proper.
4. $2^\omega \leq \omega_2$.
5. Singular Cardinal Hypothesis (SCH)
6. $\lambda^\omega = \lambda$ for any regular $\lambda \geq \omega_2$.

We study consequences on cardinal arithmetic of higher analogues of SR_{ω_1}.

Inconsistent Higher Stationary Reflection

Definition

For a regular $\kappa \geq \omega_1$, let SR_κ be the following stationary reflection principle:

For any set $W \supseteq \kappa$ and any stationary $X \subseteq \mathcal{P}_\kappa(W)$,
there is $R \subseteq W$ such that

- *$|R| = \kappa \subseteq R$,*
- *$X \cap \mathcal{P}_\kappa(R)$ is stationary in $\mathcal{P}_\kappa(R)$.*

SR_κ is inconisistent for $\kappa > \omega_1$.

Theorem (Feng-Magidor, Foreman-Magidor, Shelah-Shioya)

SR_κ fails for any regular cardinal $\kappa > \omega_1$.

On the other hand, the restriction of SR_κ to stationary sets consisting of internally approachable sets is consistent.

Internally Approachable Sets

Definition (Internally approachable sets)

Let M be a set and ρ be a regular cardinal.

- For a limit ordinal ζ, M is **internally approachable (i.a.) of length** ζ if there is a $\subseteq$-increasing sequence $\langle M_\xi \mid \xi < \zeta \rangle$ such that
  - $\bigcup_{\xi<\zeta} M_\xi = M$,
  - $\langle M_\xi \mid \xi < \zeta' \rangle \in M$ for all $\zeta' < \zeta$.
- M is **i.a.** if M is i.a. of length ζ for some ζ.
- M is **i.a. of cofinality** ρ if M is i.a. of length ζ for some ζ with $\mathrm{cof}(\zeta) = \rho$.

Definition

- $\mathrm{IA} := \{M \mid M \text{ is i.a.}\}$.
- $\mathrm{IA}_\omega := \{M \mid M \text{ is i.a. of cofinality } \omega\}$.
- $\mathrm{IA}_{>\omega} := \{M \mid M \text{ is i.a. of cofinality } > \omega\}$.

- IA_ω and $\mathrm{IA}_{>\omega}$ is somewhat similar to $\mathrm{Cof}(\omega)$ and $\mathrm{Cof}(> \omega)$, respectively.

Stationarity of IA

Fact

Suppose κ is a regular uncountable cardinal, and λ is a regular cardinal $\geq \kappa$.

- $\mathcal{P}_\kappa(\mathcal{H}_\lambda) \cap \mathrm{IA}$ is stationary in $\mathcal{P}_\kappa(\mathcal{H}_\lambda)$.
- $\mathcal{P}_{\omega_1}(\mathcal{H}_\lambda) \cap \mathrm{IA}_\omega$ is club in $\mathcal{P}_{\omega_1}(\mathcal{H}_\lambda)$.
- If $\kappa > \omega_1$, then $\mathcal{P}_\kappa(\mathcal{H}_\lambda) \cap \mathrm{IA}_{>\omega}$ is stationary in $\mathcal{P}_\kappa(\mathcal{H}_\lambda)$.
- If $\kappa > \omega_1$, then $\mathcal{P}_\kappa(\mathcal{H}_\lambda) \setminus \mathrm{IA}$ is stationary.

Restriction of Higher Stationary Reflection to IA

Let C be one of IA, IA_ω and $\mathrm{IA}_{>\omega}$.

Definition

For a regular $\kappa \geq \omega_1$, let $\mathrm{SR}_\kappa \restriction C$ be the following:

For any regular $\lambda \geq \kappa$ and any stationary $X \subseteq \mathcal{P}_\kappa(\mathcal{H}_\lambda) \cap C$, there is $R \subseteq \mathcal{H}_\lambda$ such that

- $|R| = \kappa \subseteq R$,
- $X \cap \mathcal{P}_\kappa(R)$ *is stationary in* $\mathcal{P}_\kappa(R)$.

Definition

For a regular $\kappa \geq \omega_1$, let $\mathrm{SR}^*_\kappa \restriction C$ be the following:

For any regular $\lambda \geq \kappa$ and any stationary $X \subseteq \mathcal{P}_\kappa(\mathcal{H}_\lambda) \cap C$, there is $R \subseteq \mathcal{H}_\lambda$ such that

- $|R| = \kappa \subseteq R$, *and R is i.a. of length κ*,
- $X \cap \mathcal{P}_\kappa(R)$ *is stationary in* $\mathcal{P}_\kappa(R)$.

Basic facts

Fact

1. $\mathrm{SR}_\kappa \restriction \mathrm{IA} \Leftrightarrow \mathrm{SR}_\kappa \restriction \mathrm{IA}_\omega \wedge \mathrm{SR}_\kappa \restriction \mathrm{IA}_{>\omega}$. (The same holds for $*$-versions.)
2. $\mathrm{SR}_{\omega_1} \Leftrightarrow \mathrm{SR}_{\omega_1} \restriction \mathrm{IA}$.

Theorem (Foreman-Magidor-Shelah)

For a regular uncountable cardinal κ, if a supercompact cardinal $> \kappa$ is Lévy collapsed to κ^+, then $\mathrm{SR}^*_\kappa \restriction \mathrm{IA}$ holds.

We study consequences of these higher stationary reflection principles on cardinal arithmetic.

Section 2

Higher Stationary Reflection and Cardinal Arithmetic

Power of ω and Question

- Recall that SR_{ω_1} implies that $\lambda^\omega = \lambda$ for all regular cardinal $\geq \omega_2$.
- By the same argument, we can prove the following.

Theorem 1

Let κ be a regular uncountable cardinal.
Then $\mathrm{SR}_\kappa \restriction \mathrm{IA}_\omega$ implies that $\lambda^\omega = \lambda$ for all regular $\lambda \geq \kappa^+$ (so $2^\omega \leq \kappa^+$, and SCH holds above κ).

Question

1. Does $\mathrm{SR}_\kappa \restriction \mathrm{IA}$ give any bound on 2^μ for an uncountable μ?
2. Does $\mathrm{SR}_\kappa \restriction \mathrm{IA}_{>\omega}$ give any bound on 2^ω?
3. Does $\mathrm{SR}_\kappa \restriction \mathrm{IA}_{>\omega}$ imply SCH?

We give negative answers to all these questions.

$\mathrm{SR}_\kappa \restriction \mathrm{IA}_{>\omega}$ and 2^ω

Question

Does $\mathrm{SR}_\kappa \restriction \mathrm{IA}_{>\omega}$ give any bound on 2^ω?

Answer is NO. In fact, $\mathrm{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ does not give any bound:

For a regular cardinal μ and $\nu \in \mathrm{On}$, $\mathrm{Add}(\mu, \nu)$ is the $< \mu$-closed poset adding ν-many subsets of μ, i.e. a $< \mu$-support product of ν-many copies of ${}^{<\mu}2$.

Theorem 2

Suppose κ is a regular cardinal $> \omega_1$, and $\mathrm{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ holds. Let $\nu \in \mathrm{On}$. Then $\mathrm{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ remains to hold in $V^{\mathrm{Add}(\omega,\nu)}$.

Key Lemma for Theorem 2

One difficulty to prove the preservation of SR arises from the fact that $\mathcal{P}_\kappa(W)$ changes after forcing. The following lemma allows us to avoid this difficulty.

Key Lemma

Let κ be a regular cardinal $> \omega_1$ and ν be an ordinal.
Suppose G is an $\mathrm{Add}(\omega, \nu)$-generic filter over V.

Then, in $V[G]$, for any sufficiently large regular cardinal λ, there is a club $Z \subseteq \mathcal{P}_\kappa(\mathcal{H}_\lambda)$ such that $M \cap V \in V$ for any $M \in Z \cap \mathrm{IA}_{>\omega}$.

- This lemma fails if we replace $\mathrm{IA}_{>\omega}$ with IA_ω. (Gitik)
- This lemma can be proved using the covering and approximation properties of $\mathrm{Add}(\omega, \nu)$.

Covering and Approximation Properties

Definition (Hamkins)

Let $\mathbb{P}$ be a poset, and let κ be a regular uncountable cardinal.

- $\mathbb{P}$ has **the $<\kappa$-covering property** if the following holds in $V[G]$ for any $\mathbb{P}$-generic filter G:

  For any $x \subseteq V$ with $|x| < \kappa$, there is $y \in V$ with $x \subseteq y$ and $|y|^V < \kappa$.

- $\mathbb{P}$ has **the $<\kappa$-approximation property** if for any $\mathbb{P}$-generic filter G, we have the following in $V[G]$.

  For any $x \subseteq V$, if $x \cap y \in V$ for all $y \in V$ with $|y|^V < \kappa$, then $x \in V$.

Lemma (Mitchell)

Suppose $\nu \in \mathrm{On}$. Then $\mathrm{Add}(\omega, \nu)$ has the $<\kappa$-covering and $<\kappa$-approximation properties for all regular uncountable κ.

Proof of Key Lemma

Key Lemma

Let κ be a regular cardinal $> \omega_1$ and ν be an ordinal.
Suppose G is an $\mathrm{Add}(\omega, \nu)$-generic filter over V.

Then, in $V[G]$, for any sufficiently large regular cardinal λ, there is a club $Z \subseteq \mathcal{P}_\kappa(\mathcal{H}_\lambda)$ such that $M \cap V \in V$ for any $M \in Z \cap \mathrm{IA}_{>\omega}$.

- We work in $V[G]$. Let Z be the set of all $M \in \mathcal{P}_\kappa(\mathcal{H}_\lambda)$ such that $M \prec \langle \mathcal{H}_\lambda, \in, \mathcal{H}_\lambda^V \rangle$ and $M \cap \kappa \in \kappa$.
- Suppose $M \in Z \cap \mathrm{IA}_{>\omega}$. We show $M \cap V \in V$. By the $<\omega_1$-approximation property, it suffices to show that $M \cap y \in V$ for any countable $y \in V$.
- Suppose $y \in V$ is countable. Let $\langle M_\xi \mid \xi < \zeta \rangle$ $(\mathrm{cof}(\zeta) > \omega)$ be an i.a. sequence of M. Then there is $\xi < \zeta$ such that $M \cap y \subseteq M_\xi \cap y$.
- By the $<\kappa$-covering property and the elementarity of M, there is $N \in M \cap V$ such that $M_\xi \cap \mathcal{H}_\lambda^V \subseteq N$ and $|N| < \kappa$. Note that $N \subseteq M$.
- Then $M \cap y \subseteq M_\xi \cap y \subseteq N \cap y \subseteq M \cap y$. So $M \cap y = N \cap y \in V$. $\square$

$\mathrm{SR}_\kappa \restriction \mathrm{IA}$ and 2^μ for uncountable μ

Question

Does $\mathrm{SR}_\kappa \restriction \mathrm{IA}$ give any bound on 2^μ for an uncountable μ?

Answer is NO. In fact, $\mathrm{SR}^*_\kappa \restriction \mathrm{IA}$ does not give any bound.

- It is not hard to see that $\mathrm{SR}^*_\kappa \restriction \mathrm{IA}$ does not give any bound on 2^μ for a regular $\mu \geq \kappa$.

Theorem 3

Assume GCH. Suppose κ is a regular cardinal $> \omega_1$, $\kappa \in I[\kappa]$ and $\mathrm{SR}^*_\kappa \restriction \mathrm{IA}$ holds. Let μ be a regular uncountable cardinal $< \kappa$ and ν be an ordinal.

Then $\mathrm{SR}^*_\kappa \restriction \mathrm{IA}$ remains to hold in $V^{\mathrm{Add}(\mu,\nu)}$.

Key Lemma for Theorem 3

As in Theorem 2, the following lemma allows us to avoid the change of $\mathcal{P}_\kappa(W)$.

Key Lemma

Assume GCH. Let κ be a regular cardinal $> \omega_1$ such that $\kappa \in I[\kappa]$.
Let μ be a regular uncountable cardinal $< \kappa$ and ν be an ordinal.
Suppose G is an $\mathrm{Add}(\mu, \nu)$-generic filter over V.

Then, in $V[G]$, for any sufficiently large regular cardinal λ, there is a club $Z \subseteq \mathcal{P}_\kappa(\mathcal{H}_\lambda)$ such that $M \cap V \in V$ for any $M \in Z \cap \mathrm{IA}$.

- We need some elaborations to prove the lemma for M's which are i.a. of length μ. For such M, we prove the lemma by induction on λ.

$\mathrm{SR}_\kappa \restriction \mathrm{IA}_{>\omega}$ and SCH

Question

Does $\mathrm{SR}_\kappa \restriction \mathrm{IA}_{>\omega}$ imply SCH?

Answer is NO. In fact, $\mathrm{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ does not imply SCH.

Theorem 4

Suppose κ is a regular cardinal $> \omega_1$, $\kappa \in I[\kappa]$, $2^\mu < \kappa$ for all cardinals μ with $\mu^+ < \kappa$, and $\mathrm{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ holds. Let ν be a measurable cardinal $> \kappa$ and $\mathbb{P}$ be a Prikry forcing at ν.

Then $\mathrm{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ remains to hold in $V^{\mathbb{P}}$.

- If $2^\nu > \nu^+$ in V, then $\mathrm{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ holds but SCH fails in $V^{\mathbb{P}}$.
- A Prikry forcing at ν drastically changes $\mathcal{P}_\kappa(\lambda)$ for $\lambda > \nu$, and in this case we cannot have the same key lemma as before. But we can prove some weak version.

Summary and Question

Let κ be a regular cardinal $> \omega_1$.

- $\mathrm{SR}_\kappa \restriction \mathrm{IA}_\omega$ implies that $\lambda^\omega = \lambda$ for all regular $\lambda > \kappa$ (so $2^\omega \leq \kappa^+$, and SCH holds).
- $\mathrm{SR}^*_\kappa \restriction \mathrm{IA}$ does not give any bound on 2^μ for any regular uncountable μ.
- $\mathrm{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ does not give any bound on 2^ω.
- $\mathrm{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ does not imply SCH.

I do not know the answer of the following question:

Question

Does $\mathrm{SR}_\kappa \restriction \mathrm{IA}_{>\omega}$ or $\mathrm{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ imply SCH at singular cardinals of uncountable cofinality?