Definability of maximal families of reals in forcing extensions

Jonathan Schilhan

Institute of Mathematics, University of Vienna

RIMS Workshop, 16.11.2020

generous support through FWF Project Y1012-N35

Hypergraphs

Definition

A *hypergraph* on a set X is a collection E (the edges) of finite non-empty subsets of X, i.e. $E \subseteq [X]^{<\omega} \setminus \{\emptyset\}$. We say that $Y \subseteq X$ is *E-independent* if $[Y]^{<\omega} \cap E = \emptyset$. Y is *maximal E-independent* if Y is maximal under inclusion as an E-independent subset of X.

If X is a Polish space, then $[X]^{<\omega}$ also has a natural Polish topology and we can study definable hypergraphs E and definable maximal E-independent sets.

Fact

In L, every analytic hypergraph on a Polish space X has a $\mathbf{\Delta}^1_2$ maximal independent set.

Graphs

Theorem (Schrittesser 2016)

After forcing with a csi of Sacks forcing over L, every analytic (2-dimensional hyper)graph in a Polish space has a $\mathbf{\Delta}^1_2$ maximal independent set.

Example

In an extension of L by a csi of Sacks forcing, every analytic equivalence relation has a Δ^1_2 transversal.

This is wrong for the countable support product. There is no $OD(\mathbb{R})$ transversal of E_1 after an uncountably supported csp of Sacks forcing.

Examples

Example (Ultrafilter)

Let $E_u := \{A \in [\mathcal{P}(\omega)]^{<\omega} : \bigcap A \text{ is finite}\}$. Then an ultrafilter is a maximal E_u-independent set.

Question (S. 2019)

Is it consistent that $\mathfrak{u} > \aleph_1$, while there is a Δ^1_2 ultrafilter?
Can we destroy all ground model ultrafilters while preserving a Δ^1_2 definition?

Examples

Example (Maximal independent families)

$Y \subseteq \mathcal{P}(\omega)$ is an independent family if for all finite disjoint $A, B \subseteq Y$, $\bigcap_{x \in A} x \cap \bigcap_{x \in B} \omega \setminus x$ is infinite. Letting

$$E_i := \{A \dot\cup B \in [\mathcal{P}(\omega)]^{<\omega} : \bigcap_{x \in A} x \cap \bigcap_{x \in B} \omega \setminus x \text{ is finite}\}$$

an independent family is an E_i-independent set.

Question (Brendle, Fischer, Khomskii 2019)

Is it consistent that $\mathfrak{i} > \aleph_1$, while there is a Π^1_1 maximal independent family?
Can we destroy all ground model maximal independent families while preserving a Π^1_1 definition?

Examples

Example (Hamel basis)

Let $E_h := \{A \in [\mathbb{R}]^{<\omega} : A \text{ is linearly dependent over } \mathbb{Q}\}$. Then a Hamel basis is a maximal E_h-independent set.

Every Hamel basis has size $2^{\aleph_0}$. This is reflected by the fact that adding a single real destroys every ground model Hamel basis.

Question

Is it consistent that $\neg$ CH, while there is a Δ^1_2 Hamel basis?
Can we destroy all ground model Hamel bases (i.e. add a new real) while preserving a Δ^1_2 definition?

Splitting reals

Definition

Let $x, y \in [\omega]^\omega$. Then x splits y if $|x \cap y| = \omega$ and $|y \setminus x| = \omega$. x is called a splitting real over V if for every $y \in V \cap [\omega]^\omega$, x splits y.

Adding splitting reals over a model V, destroys all ultrafilters, maximal independent family (and Hamel bases) in V and consequently increases $\mathfrak{u}, \mathfrak{i}$.

Classical forcing notions adding splitting reals are Cohen, Random, Silver forcing and forcings adding dominating reals. But they all fail in preserving Δ^1_2 witnesses.

Splitting forcing

Definition (Splitting forcing)

A set $A \subseteq 2^{<\omega}$ is called *fat* if there is $m = m(A) \in \omega$ so that for every $n \geq m$, $i \in 2$, there is $s \in A$ so that $s(n) = i$.
Let $T \subseteq 2^{<\omega}$ be a perfect tree. Then T is a *splitting tree* if for every $s \in T$, T_s is fat.
Splitting forcing $\mathbb{SP}$ consists of all splitting trees ordered by inclusion ($T \leq S$ iff $T \subseteq S$), as usual.

Fact

- $\mathbb{SP}$ *adds a generic splitting real* $x_G \in 2^\omega (\cong \mathcal{P}(\omega))$,
- $\mathbb{SP}$ *is proper (Axiom A),*
- $\mathbb{SP}$ *is* ω^ω*-bounding,*
- $V^{\mathbb{SP}}$ *is a minimal extension of* V.

The main result

Theorem (S. 2020)

After forcing with a csi or a finite product of Sacks or splitting forcing over L, every analytic hypergraph in a Polish space has a Δ^1_2 maximal independent set.

Corollary

It is consistent that there is a Π^1_1 mif, a Δ^1_2 ultrafilter and a Δ^1_2 Hamel basis while $\aleph_1 < \mathfrak{i}, \mathfrak{u}, \mathfrak{c}$.

Note:

Theorem (Brendle, Khomski, Fischer 2019)

The existence of a Σ^1_2 maximal independent family is equivalent to the existence of a Π^1_1 maximal independent family.

Borelized cardinal invariants

Definition

$\mathfrak{u}_B = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathbf{\Delta}^1_1 \text{ and } \bigcup \mathcal{B} \text{ is an ultrafilter}\}$
$\mathfrak{i}_B = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathbf{\Delta}^1_1 \text{ and } \bigcup \mathcal{B} \text{ is a max. ind. family}\}$
similarly: $\mathfrak{u}_{cl} = \min \ldots$, $\mathfrak{i}_{cl} = \min \ldots$

Observation

$\mathfrak{u}_B \leq \mathfrak{u}_{cl} \leq \mathfrak{u}$, $\mathfrak{i}_B \leq \mathfrak{i}_{cl} \leq \mathfrak{i}$.
If there is a $\mathbf{\Delta}^1_2$ ultrafilter/m.i.f then $\mathfrak{u}_B/\mathfrak{i}_B = \omega_1$.

Corollary

It is consistent that $\mathfrak{u}_B, \mathfrak{i}_B < \mathfrak{u}, \mathfrak{i}$.

Theorem (S. 2020)

(ZFC) $\mathfrak{d} \leq \mathfrak{i}_{cl}$.

Recall: $\mathfrak{d} \leq \mathfrak{i}$.

Other applications

Corollary of the construction

There is a (Δ^1_2) P-point after iterating $\mathbb{SP}$ over L.

Corollary

ZF+DC+"every analytic hypergraph has a maximal independent set"+"there is no well-order of the reals" is consistent relative to ZF.

Question (Pincus, Prikry 1975)

(ZF) Does the existence of a Hamel basis imply a well-order of the reals?

This has been answered before by [Beriashvili, Schindler, Wu, Yu 2018] (without LC) and by [Larson, Zapletal 2017] using an inaccessible.

Questions

Question

Does the main result hold true for the csi of Miller forcing?

Remark: It fails for finite products of Miller forcing ($\mathbb{M}^3$ adds a Cohen real).

This would yield a model of $\omega_1 = \mathfrak{i}_B < \mathfrak{d} = \mathfrak{i}_{cl} = \mathfrak{i} = \omega_2$.

Question (Brendle, Khomskii, Fischer 2019)

Is $\mathfrak{i}_B < \mathfrak{i}_{cl}$ consistent?

Thank you!