Hereditary Interval Algebras and Cardinal Characteristics of the Continuum

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Interval Algebras.

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- A subinterval algebra is a Boolean algebra which can be embedded into an interval algebra.

Basic Facts.

- If $\mathcal{B} \twoheadrightarrow \mathcal{A}$ is a surjective homomorphism, and $\mathcal{B}$ is an interval algebra, then $\mathcal{A}$ is also an interval algebra.

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Example 2

The algebra of finite and cofinite subsets of ω_1, is a subinterval algebra which is not an interval algebra.

Hereditary Interval Algebras.

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Theorem 4 (Bekkali, Todorcevic, 2015)

Every hereditary interval algebra is σ-centered.

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Theorem 5 (Nikiel, Purisch, Treybig, 1998)

There is a σ-centered interval algebra of size contiuum which is not hereditary.

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Theorem 5 (Nikiel, Purisch, Treybig, 1998)

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Definition 6

Let $\mathfrak{hia}$ be

$$\min\{|B| : B \text{ is a nonhereditary } \sigma\text{−centered interval algebra}\}$$

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Is $\mathfrak{b} = \mathfrak{hia}$?

Main Results.

Theorem 8 (Hrušák, Ramos-García, M-R)

It is consistent, with ZFC, that $\mathfrak{b} < \mathfrak{hia}$.

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Theorem 9 (Hrušák, Ramos-García, M-R)

There is, in ZFC, an uncountable hereditary interval algebra.

Definition 10

Let $(L, \leq)$ be a linearly ordered set, $D \subseteq L$ and $f \in 2^D$. Let $(L, \leq, \tau_f)$ denote the generalised order space whose underlying set is L, and whose topology is generated by the subbase:

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- $(-\infty, d]$ for $d \in D$ and $f(d) = 0$;

Combinatorial Reformulation of $\mathfrak{hia}$.

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- $[x, \infty)$ for $d \in D$ and $f(d) = 1$.

Lower bound.

Theorem 11 (Hrušák, Ramos-García, M-R)

$\mathfrak{hia} = \nu$ *where*
$\nu = \min\{|X| : \mathbb{Q} \subseteq X \subseteq \mathbb{R}, \mathbb{Q}$ *is not a* G_δ *in* $(X, \leq, \tau_f)$ *for any* $f \in 2^{\mathbb{Q}}\}$

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- Finally take $A(X)/\sim$ where we identify $(q,0) \sim (q,2)$ for any $q \in \mathbb{Q} \cap (0,1)$.

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Recall that a pseudotree is a partially ordered set T so that the set $\{y \in T : y \leq x\}$ is linearly ordered for all $x \in T$.

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Theorem 14 (Heindorff, 1997)

Every subinterval algebra is isomorphic to one of the form $\mathcal{B}(T)$ for some pseudo-tree T.

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  - ⋆ and $\prec_\infty$ is, roughly speaking, the limit of the orders $\prec_n$.

Comparison with classical cardinal invariants.

Lemma 15

$\mathfrak{b} \leq \mathfrak{hia} \leq \min\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$.

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- Construct a sequence $(q_s : s \in 2^{<\omega})$, and a Cantor's scheme of closed intervals $(I_s : s \in 2^{<\omega})$ such that:
  - ⋆ $I_s = [-\epsilon_s + q_s, q_s + \epsilon_s] \subset (-\frac{1}{g(q_s)} + q_s, q_s + \frac{1}{g(q_s)})$;
  - ⋆ $I_{s^\frown 0} \subset (-\frac{1}{g(q_s)} + q_s, q_s)$ and $I_{s^\frown 1} \subset (q_s, q_s + \frac{1}{g(q_s)})$;
  - ⋆ Let $X_g = \{x_\alpha : \alpha < \omega_1\}$ be any subset such that for any $x \in X_g$ there is an $s_x \in 2^\omega$ so that $x \in \bigcap I_{s_x \restriction n}$.
  - ⋆ Set $a_\alpha = \{q \in \mathbb{Q} : 0 < q - x_\alpha < \frac{1}{g(q)}\}$ and $b_\alpha = \{q \in \mathbb{Q} : 0 < x_\alpha - q < \frac{1}{g(q)}\}$
  - ⋆ $(a_\alpha, b_\alpha : \alpha < \omega_1)$ form a Luzin gap.
- Set $X = \bigcup_{g \in \mathcal{F}} X_g$;
- If $\mathbb{Q}$ is a G_δ in the generalised order space $(X \cup \mathbb{Q}, \leq, \tau_f)$ for some f, then one of the gaps can be separated.

Consistency results.

Lemma 16 (Brendle, 95)

Suppose $\mathbb{P}$ *is a ccc poset,* $h \colon \mathbb{P} \to \omega$ *is a height function (order reversing) on* $\mathbb{P}$*, and* $(\mathbb{P}, h)$ *is soft then any unbounded family of functions in* $\omega^\omega \cap V$ *is still unbounded in* $V[G]$*, where* G *is* $\mathbb{P}$*-generic over* V*. Moreover, this is preserved under finite support iteration of ccc posets.*

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Given any uncountable regular cardinal κ, it is consistent that $\mathfrak{hia} = k$ and $\mathfrak{b} = \aleph_1$.

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Given any uncountable regular cardinal κ, it is consistent that $\mathfrak{hia} = k$ and $\mathfrak{b} = \aleph_1$.

- We construct a σ-centered poset $\mathbb{P}$, and a height function h so that:
  - ⋆ The pair $(\mathbb{P}, h)$ is soft,
  - ⋆ If G is a $\mathbb{P}$-generic filter over V, then $\Vdash_{\mathbb{P}}$ "$\mathbb{Q}$ *is* G_δ *in* $\mathbb{R} \cap V$".

ZFC results.

Let $(a_\alpha, b_\alpha : \alpha < \omega_1)$ be a Hausdorff gap, and assume, that $L = \{a_\alpha : \alpha < \omega_1\} \cup \{b_\alpha : \alpha < \omega_1\}$ is dense in 2^ω.

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Theorem 18 (Hrušák, Ramos-García, M-R)

The interval algebra $\mathcal{B}(L)$ is hereditary.

Thank you for your attention!